

Correlation functions for zeros of a Gaussian power series and Pfaffians

SHO MATSUMOTO AND TOMOYUKI SHIRAI

Abstract

We show that the zeros of the random power series with i.i.d. real Gaussian coefficients form a Pfaffian point process. We also show that the product moments for absolute values and signatures of the power series can also be expressed by Pfaffians.

1 Introduction

Zeros of Gaussian processes have attracted much attention for many years both from theoretical and practical points of view. The first significant contribution to this study was made by Paley-Wiener [18]. They computed the expectation of the number of zeros of (translation invariant) analytic Gaussian processes on a strip in the complex plane that are defined as Wiener integrals. Their work was motivated by papers due to Bohr-Jessen [1, 2] on almost periodic functions in the complex domain arising from Riemann's zeta function. Kac gives an explicit expression of the probability density function of real zeros of a random polynomial

$$f_n(t) = \sum_{k=0}^n a_k t^k$$

with i.i.d. real standard Gaussian coefficients $\{a_k\}_{k=0}^n$ and obtains a precise asymptotics of the numbers of real zeros as $n \rightarrow \infty$ [11]. Rice also obtained similar formulas for the zeros of random Fourier series with Gaussian coefficients in the theory of filtering [19]. Their results have been extended in various ways (e.g. [5, 14, 20]) and generalizations of their formulas are sometimes called the Kac-Rice formulas. A recent remarkable result on zeros of Gaussian processes is that the complex Gaussian process $f_{\mathbb{C}}(z) := \sum_{k=0}^{\infty} \zeta_k z^k$ with i.i.d. complex standard Gaussian coefficients form a determinantal point process on the open unit disk \mathbb{D} associated with the Bergman kernel $K(z, w) = \frac{1}{(1-z\bar{w})^2}$, which was found by Peres-Virág [17]. Krishnapur extended this result to the zeros of determinant of the power series with coefficients being i.i.d. Ginibre matrices [13].

In the present paper, we deal with the Gaussian power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad (1.1)$$

where $\{a_k\}_{k=0}^{\infty}$ are i.i.d. *real* standard Gaussian random variables. The radius of convergence of f is almost surely 1, and the set of the zeros of f forms a point process on the open unit disc \mathbb{D} as that of $f_{\mathbb{C}}$ does. The primary difference between f and $f_{\mathbb{C}}$ comes from the fact that $f(z)$ is a real Gaussian process when the parameter z is restricted on $(-1, 1)$ and each realization of $f(z)$ has symmetry with respect to the complex conjugation so that there appear both real zeros and complex ones in conjugate pairs. The point process of real zeros and that of complex zeros are singular each other.

Our main purpose is to show that both correlation functions for real zeros and complex zeros of f are given by Pfaffians, i.e., they form Pfaffian point processes on $(-1, 1)$ and \mathbb{D} , respectively. The most known examples of Pfaffian point processes appeared as random eigenvalues of the Gaussian orthogonal/symplectic ensembles. Real and complex eigenvalues of the *real* Ginibre ensemble are also proved to be Pfaffian point processes on \mathbb{R} and \mathbb{C} , respectively [3, 7]. Recently, it is shown that the particle positions of instantly coalescing (or annihilating) Brownian motions on the real line under the maximal entrance law form a Pfaffian point process on \mathbb{R} [21], which is closely related to the real Ginibre ensemble. Our result on correlation functions of zeros of f is added to the list of Pfaffian point processes, which is also obtained independently in [6] via random matrix theory. Here we will give a direct proof by using Hammersley's formula for correlation functions of zeros of Gaussian analytic functions and a Pfaffian-Hafnian identity due to Ishikawa-Kawamuko-Okada [10]. This is a similar way that was taken in [17] to prove that the zeros of $f_{\mathbb{C}}$ form a determinantal point process, and in the process of our calculus for real zero correlations, we obtain new Pfaffian formulas for a real Gaussian process. The family $\{f(t)\}_{-1 < t < 1}$ can be regarded as a centered real Gaussian process with covariance kernel $(1 - st)^{-1}$. We show that, for any $-1 < t_1, t_2, \dots, t_n < 1$, both the moments of absolute values $E[|f(t_1)f(t_2) \cdots f(t_n)|]$ and those of signatures $E[\text{sgn } f(t_1) \cdots \text{sgn } f(t_n)]$ are also given by Pfaffians. We stress that it should be *surprising* because such combinatorial formulas cannot be expected for general centered Gaussian processes. These are special features for the Gaussian process with covariance kernel $(1 - st)^{-1}$.

The paper is organized as follows. In Section 2, we state our main results for correlations of real and complex zeros of f (Theorems 1 and 6), and we give new product moment formulas for absolute values and signatures of f (Theorems 4 and 5). Also we observe negative correlation property of real and complex zeros by showing negative correlation inequalities for 2-correlation functions. The asymptotics of the number of real zeros inside intervals growing to $(-1, 1)$ is also shown. In Section 3, we recall the well-known Cauchy's determinant formula and the Wick formula for product moments of Gaussian random variables. In Section 4, after we show an identity in law for f and f' given that

f is vanishing at some points, we give a preliminary version of Pfaffian formulas (Proposition 4.4) for the derivation of the expectation of product of sign functions. In Sections 5, 6 and 7, we give the proofs of our results stated in Section 2.

2 Results

2.1 Pfaffians

Our main results will be described by using Pfaffians. Let us recall the definition of Pfaffians. For a $2n \times 2n$ skew symmetric matrix $B = (b_{ij})_{i,j=1}^{2n}$, the Pfaffian of B is defined by

$$\text{Pf}(B) = \sum_{\eta} (\text{sgn } \eta) b_{\eta(1)\eta(2)} b_{\eta(3)\eta(4)} \cdots b_{\eta(2n-1)\eta(2n)},$$

summed over all permutations η on $\{1, 2, \dots, 2n\}$ satisfying $\eta(2i-1) < \eta(2i)$ ($i = 1, 2, \dots, n$) and $\eta(1) < \eta(3) < \cdots < \eta(2n-1)$. Here $\text{sgn } \eta$ is the signature of η . For example,

$$\text{Pf}(B) = b_{11} \quad \text{if } n = 1 \quad \text{and} \quad \text{Pf}(B) = b_{12}b_{34} - b_{13}b_{24} + b_{14}b_{23} \quad \text{if } n = 2. \quad (2.1)$$

For an upper-triangular array $A = (a_{ij})_{1 \leq i < j \leq 2n}$, we define the Pfaffian of A as that of the skew-symmetric matrix $B = (b_{ij})_{i,j=1}^{2n}$, each entry of which is $b_{ij} = -b_{ji} = a_{ij}$ if $i < j$ and $b_{ii} = 0$.

2.2 Notation

For $-1 < s, t < 1$, we define the functions

$$\sigma(s, t) = \frac{1}{1-st}, \quad c(s, t) = \frac{\sigma(s, t)}{\sqrt{\sigma(s, s)\sigma(t, t)}} = \frac{\sqrt{(1-s^2)(1-t^2)}}{1-st} \quad (2.2)$$

and the skew symmetric matrix kernel \mathbb{K} by

$$\mathbb{K}(s, t) = \begin{pmatrix} \mathbb{K}_{11}(s, t) & \mathbb{K}_{12}(s, t) \\ \mathbb{K}_{21}(s, t) & \mathbb{K}_{22}(s, t) \end{pmatrix}$$

with

$$\begin{aligned} \mathbb{K}_{11}(s, t) &= \frac{s-t}{\sqrt{(1-s^2)(1-t^2)}(1-st)^2}, & \mathbb{K}_{12}(s, t) &= \sqrt{\frac{1-t^2}{1-s^2}} \frac{1}{1-st}, \\ \mathbb{K}_{21}(s, t) &= -\sqrt{\frac{1-s^2}{1-t^2}} \frac{1}{1-st}, & \mathbb{K}_{22}(s, t) &= \text{sgn}(t-s) \arcsin c(s, t), \end{aligned}$$

where $\text{sgn}(t) = |t|/t$ for $t \neq 0$ and $\text{sgn}(t) = 0$ for $t = 0$. Note that $\mathbb{K}_{12}(s, t) = -\mathbb{K}_{21}(t, s)$ and

$$\mathbb{K}(s, t) = \begin{pmatrix} \frac{\partial^2}{\partial s \partial t} \mathbb{K}_{22}(s, t) & \frac{\partial}{\partial s} \mathbb{K}_{22}(s, t) \\ \frac{\partial}{\partial t} \mathbb{K}_{22}(s, t) & \mathbb{K}_{22}(s, t) \end{pmatrix}. \quad (2.3)$$

We denote by $\text{Pf}(\mathbb{K}(t_i, t_j))_{i,j=1}^n$ the Pfaffian of the $2n \times 2n$ skew symmetric matrix

$$\begin{pmatrix} \mathbb{K}(t_1, t_1) & \mathbb{K}(t_1, t_2) & \dots & \mathbb{K}(t_1, t_n) \\ \mathbb{K}(t_2, t_1) & \mathbb{K}(t_2, t_2) & \dots & \mathbb{K}(t_2, t_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{K}(t_n, t_1) & \mathbb{K}(t_n, t_2) & \dots & \mathbb{K}(t_n, t_n) \end{pmatrix}.$$

Throughout this paper, \mathfrak{X}_n denotes the set of all sequences $\mathbf{t} = (t_1, \dots, t_n)$ of n distinct real numbers in the interval $(-1, 1)$.

2.3 Real zero correlations

Our first theorem states that the real zero distribution of f forms a Pfaffian point process.

Theorem 1. *Let ρ_n be the n -point correlation function of real zeros of the Gaussian power series defined in (1.1). Then*

$$\rho_n(t_1, \dots, t_n) = \pi^{-n} \text{Pf}(\mathbb{K}(t_i, t_j))_{i,j=1}^n \quad (-1 < t_1, \dots, t_n < 1).$$

From the explicit expression of correlation functions, for example, the first two correlations are given as follows:

$$\begin{aligned} \rho_1(s) &= \pi^{-1} \mathbb{K}_{12}(s, s) \\ \rho_2(s, t) &= \pi^{-2} \{ \mathbb{K}_{12}(s, s) \mathbb{K}_{12}(t, t) - \mathbb{K}_{11}(s, t) \mathbb{K}_{22}(s, t) + \mathbb{K}_{12}(s, t) \mathbb{K}_{21}(s, t) \}, \end{aligned} \quad (2.4)$$

from which we easily see that

$$\rho_1(t) = \frac{1}{\pi(1-t^2)}, \quad \rho_2(s, t) = \frac{1}{2\pi(1-s^2)^3} |t-s| + O(|t-s|^2)$$

as $t \rightarrow s$. The first correlation is observed by Kac and many others although Kac considered the random polynomial with i.i.d. real Gaussian coefficients. The second asymptotic expression means that the real zeros of f are repulsive each other as expected. Moreover, we can show that the 2-correlation is negatively correlated.

Corollary 2. *Let $R(s, t) = \frac{\rho_2(s, t)}{\rho_1(s)\rho_1(t)}$ be the normalized 2-point correlation function. Then, $R(s, s) = 0$, $R(s, \pm 1) = 1$ and $R(s, t)$ is strictly increasing (resp. decreasing) for $t \in [s, 1]$ (resp. $t \in [-1, s]$). In particular, $\rho_2(s, t) \leq \rho_1(s)\rho_1(t)$ for every $s, t \in (-1, 1)$.*

By using (2.4), we can also compute the mean and variance of the number of points inside $[-r, r]$.

Corollary 3. *Let N_r be the number of real zeros in the interval $[-r, r]$ for $0 < r < 1$. Then,*

$$EN_r = \frac{1}{\pi} \log \frac{1+r}{1-r}, \quad \text{Var } N_r = 2 \left(1 - \frac{2}{\pi}\right) EN_r + O(1)$$

as $r \rightarrow 1$.

Remark 2.1. The kernel \mathbb{K} in Theorem 1 is not determined uniquely. For example, we can replace \mathbb{K} by \mathbb{K}' , which is defined by

$$\begin{aligned} \mathbb{K}'_{11}(s, t) &= \frac{s-t}{(1-st)^2}, & \mathbb{K}'_{12}(s, t) &= -\mathbb{K}'_{21}(t, s) = \frac{1}{1-st}, \\ \mathbb{K}'_{22}(s, t) &= \frac{\text{sgn}(t-s)}{\sqrt{(1-s^2)(1-t^2)}} \arcsin c(s, t). \end{aligned}$$

In fact, if we set

$$Q(s, t) = \delta_{st} \begin{pmatrix} \sqrt{1-t^2} & 0 \\ 0 & \frac{1}{\sqrt{1-t^2}} \end{pmatrix}$$

then $(Q(t_i, t_j))_{i,j=1}^n \cdot (\mathbb{K}(t_i, t_j))_{i,j=1}^n \cdot (Q(t_i, t_j))_{i,j=1}^n = (\mathbb{K}'(t_i, t_j))_{i,j=1}^n$, and therefore two Pfaffians associated with \mathbb{K} and \mathbb{K}' coincide from the following well-known identity: for any $2n \times 2n$ matrix A and $2n \times 2n$ skew symmetric matrix B , $\text{Pf}(ABA^t) = (\det A)(\text{Pf } B)$.

2.4 Pfaffian formulas for a real Gaussian process

As corollaries of the proof of Theorem 1, we obtain Pfaffian expressions for averages of $|f(t_1) \cdots f(t_n)|$ and $\text{sgn } f(t_1) \cdots \text{sgn } f(t_n)$.

Theorem 4. *Let f be the Gaussian power series defined in (1.1). For $\mathbf{t} = (t_1, \dots, t_n) \in \mathfrak{X}_n$, we have*

$$E[|f(t_1)f(t_2) \cdots f(t_n)|] = \left(\frac{2}{\pi}\right)^{n/2} (\det \Sigma(\mathbf{t}))^{-\frac{1}{2}} \text{Pf}(\mathbb{K}(t_i, t_j))_{i,j=1}^n,$$

where $\Sigma(\mathbf{t})$ is the $n \times n$ positive-definite symmetric matrix given by

$$\Sigma(\mathbf{t}) = (\sigma(t_i, t_j))_{i,j=1}^n, \quad \sigma(s, t) = \frac{1}{1-st}.$$

Theorem 5. *Let f be the Gaussian power series defined in (1.1). For $(t_1, \dots, t_{2n}) \in \mathfrak{X}_{2n}$, we have*

$$\begin{aligned} & E[\operatorname{sgn} f(t_1) \operatorname{sgn} f(t_2) \cdots \operatorname{sgn} f(t_{2n})] \\ &= \left(\frac{2}{\pi}\right)^n \prod_{1 \leq i < j \leq 2n} \operatorname{sgn}(t_j - t_i) \cdot \operatorname{Pf}(\mathbb{K}_{22}(t_i, t_j))_{i,j=1}^{2n}. \end{aligned}$$

In particular, if $-1 < t_1 < t_2 < \cdots < t_{2n} < 1$, then

$$E[\operatorname{sgn} f(t_1) \operatorname{sgn} f(t_2) \cdots \operatorname{sgn} f(t_{2n})] = \left(\frac{2}{\pi}\right)^n \operatorname{Pf}(\arcsin c(t_i, t_j))_{1 \leq i < j \leq 2n}, \quad (2.5)$$

where $c(s, t)$ is defined in (2.2).

We note that it is easy to observe $E[\operatorname{sgn} f(t_1) \cdots \operatorname{sgn} f(t_n)] = 0$ when n is odd. More generally, if (X_1, \dots, X_n) is a centered real Gaussian vector, then $E[\operatorname{sgn} X_1 \cdots \operatorname{sgn} X_n] = 0$ for n odd. Moreover, the formula (2.5) can be rewritten as

$$E[\operatorname{sgn} f(t_1) \operatorname{sgn} f(t_2) \cdots \operatorname{sgn} f(t_{2n})] = \operatorname{Pf}(E[\operatorname{sgn} f(t_i) \operatorname{sgn} f(t_j)])_{1 \leq i < j \leq 2n}.$$

As explained later, the Wick formula (3.6) provides us a similar formula for products of real Gaussian random variables, however, such neat formulas for $E[|X_1 X_2 \cdots X_n|]$ are not known for general n except the cases with $n = 2, 3$ ([15, 16]). Similarly, there is no known formula for $E[\operatorname{sgn} X_1 \operatorname{sgn} X_2 \cdots \operatorname{sgn} X_{2n}]$ except the $n = 2$ case

$$E[\operatorname{sgn} X_1 \operatorname{sgn} X_2] = \frac{2}{\pi} \arcsin \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}},$$

where $\sigma_{ij} = E[X_i X_j]$ for $i, j = 1, 2$. Theorem 4 and Theorem 5 state that the moments $E[|X_1 \cdots X_n|]$ and $E[\operatorname{sgn} X_1 \cdots \operatorname{sgn} X_n]$ have Pfaffian expressions if the covariance matrix of the real Gaussian vector (X_1, \dots, X_n) is of the form $((1 - t_i t_j)^{-1})_{i,j=1}^n$.

2.5 Complex zero correlations

The complex zero distribution also forms a Pfaffian point process. Put $\mathbb{D}_+ = \{z \in \mathbb{C} \mid |z| < 1, \Im z > 0\}$.

Theorem 6. *Let ρ_n^c be the n -point correlation function for complex zeros of f . For $z_1, z_2, \dots, z_n \in \mathbb{D}_+$,*

$$\rho_n^c(z_1, \dots, z_n) = \frac{1}{(\pi\sqrt{-1})^n} \prod_{j=1}^n \frac{1}{|1 - z_j^2|} \cdot \operatorname{Pf}(\mathbb{K}^c(z_i, z_j))_{i,j=1}^n,$$

where $\mathbb{K}^c(z, w)$ is 2×2 matrix kernel

$$\mathbb{K}^c(z, w) = \begin{pmatrix} \frac{z-w}{(1-zw)^2} & \frac{z-\bar{w}}{(1-z\bar{w})^2} \\ \frac{\bar{z}-w}{(1-\bar{z}w)^2} & \frac{\bar{z}-\bar{w}}{(1-\bar{z}\bar{w})^2} \end{pmatrix}.$$

For example, the one-point correlation is

$$\rho_1^c(z) = \frac{|z - \bar{z}|}{\pi|1 - z^2|(1 - |z|^2)^2}$$

and

$$\rho_2^c(z, w) = \rho_1^c(z)\rho_1^c(w) + \frac{1}{\pi^2|1 - z^2||1 - w^2|} \left(\left| \frac{z - w}{1 - zw} \right|^2 - \left| \frac{z - \bar{w}}{1 - z\bar{w}} \right|^2 \right).$$

It is easy to verify that $\rho_2^c(z, w) < \rho_1^c(z)\rho_1^c(w)$ for $z, w \in \mathbb{D}_+$, which implies negative correlation as well as the case of real zeros.

As we mentioned, Theorem 1 and Theorem 6 are obtained independently in [6] via random matrix theory, but Theorem 4 and Theorem 5 are new.

3 Cauchy's determinants and Wick formula

In this short section, we review Cauchy's determinants and Wick formula, which are essential throughout this paper.

3.1 Cauchy's determinant and its variations

The following identity for a determinant, the so-called Cauchy determinant identity, is well known in combinatorics. See, e.g., [4, Proposition 4.2.3].

$$\det \left(\frac{1}{1 - x_i y_j} \right)_{i,j=1}^n = \frac{\prod_{1 \leq i < j \leq n} (x_i - x_j)(y_i - y_j)}{\prod_{i=1}^n \prod_{j=1}^n (1 - x_i y_j)}. \quad (3.1)$$

Here the x_i, y_j are formal variables, but we will assume that they are complex numbers in \mathbb{D} when we apply formulas contained in this subsection. For each $i = 1, 2, \dots, n$, we define $q_i(\mathbf{x}) = q_i(x_1, \dots, x_n)$ by

$$q_i(\mathbf{x}) = \frac{1}{1 - x_i^2} \prod_{\substack{1 \leq k \leq n \\ k \neq i}} \frac{x_i - x_k}{1 - x_i x_k}. \quad (3.2)$$

Using (3.1), we have

$$q_1(\mathbf{x})q_2(\mathbf{x}) \cdots q_n(\mathbf{x}) = (-1)^{n(n-1)/2} \det \left(\frac{1}{1 - x_i x_j} \right)_{i,j=1}^n. \quad (3.3)$$

Recall the definition of Hafnians, which are sign-less analogs of Pfaffians. For a $2n \times 2n$ symmetric matrix $A = (a_{ij})_{i,j=1}^{2n}$, the Hafnian of A is defined by

$$\text{Hf } A = \sum_{\eta} a_{\eta(1)\eta(2)} a_{\eta(3)\eta(4)} \cdots a_{\eta(2n-1)\eta(2n)}, \quad (3.4)$$

summed over all permutations η on $\{1, 2, \dots, 2n\}$ satisfying $\eta(2i-1) < \eta(2i)$ ($i = 1, 2, \dots, n$) and $\eta(1) < \eta(3) < \dots < \eta(2n-1)$.

A Pfaffian version of Cauchy's determinant identity is Schur's Pfaffian identity:

$$\text{Pf} \left(\frac{x_i - x_j}{1 - x_i x_j} \right)_{i,j=1}^{2n} = \prod_{1 \leq i < j \leq 2n} \frac{x_i - x_j}{1 - x_i x_j}.$$

The following formula due to Ishikawa-Kawamuko-Okada [10] will be an important factor in our proofs of theorems.

$$\prod_{1 \leq i < j \leq 2n} \frac{x_i - x_j}{1 - x_i x_j} \cdot \text{Hf} \left(\frac{1}{1 - x_i x_j} \right)_{i,j=1}^{2n} = \text{Pf} \left(\frac{x_i - x_j}{(1 - x_i x_j)^2} \right)_{i,j=1}^{2n}. \quad (3.5)$$

3.2 Wick formula

We recall the method for computations of expectations of polynomials in real Gaussian random variables. Let Y_1, \dots, Y_n be linear combinations of centered real Gaussian random variables. Then $E[Y_1 \cdots Y_n] = 0$ if n is odd, and

$$E[Y_1 Y_2 \cdots Y_n] = \text{Hf}(E[Y_i Y_j])_{i,j=1}^n \quad (3.6)$$

if n is even. For example, $E[Y_1 Y_2 Y_3 Y_4] = E[Y_1 Y_2]E[Y_3 Y_4] + E[Y_1 Y_3]E[Y_2 Y_4] + E[Y_1 Y_4]E[Y_2 Y_3]$. See, e.g., survey [24] for details.

4 Derivations of sign moments

In this section, we provide a preliminary version of Pfaffian formulas for Theorem 5.

4.1 Derivations for real Gaussian processes

A derivation for sign moments of real Gaussian processes is given by a conditional expectation in the following way.

Lemma 4.1. *Let $\{X(t)\}_{-1 < t < 1}$ be a smooth real Gaussian process with covariance kernel K . Let $(t_1, t_2, \dots, t_n, s_1, s_2, \dots, s_m) \in \mathfrak{X}_{n+m}$ and suppose that $\det K(\mathbf{t}) = \det(K(t_i, t_j))_{i,j=1}^n$ does not vanish. Then,*

$$\begin{aligned} & \frac{\partial^n}{\partial t_1 \partial t_2 \cdots \partial t_n} E[\text{sgn } X(t_1) \cdots \text{sgn } X(t_n) \text{sgn } X(s_1) \cdots \text{sgn } X(s_m)] \\ &= \left(\frac{2}{\pi} \right)^{n/2} E[X'(t_1) \cdots X'(t_n) \text{sgn } X(s_1) \cdots \text{sgn } X(s_m) \mid X(t_1) = \cdots = X(t_n) = 0] \\ & \quad \times (\det K(\mathbf{t}))^{-1/2}. \end{aligned}$$

Proof. We will give a heuristic proof. The derivation of $\operatorname{sgn} t$ is $\frac{\partial}{\partial t} \operatorname{sgn} t = 2\delta_0(t)$, where $\delta_0(t)$ is Dirac's delta function at 0. Hence, if we abbreviate as $Y(\mathbf{s}) = \operatorname{sgn} X(s_1) \cdots \operatorname{sgn} X(s_m)$, then

$$\begin{aligned} & \frac{\partial^n}{\partial t_1 \partial t_2 \cdots \partial t_n} E[\operatorname{sgn} X(t_1) \cdots \operatorname{sgn} X(t_n) \cdot Y(\mathbf{s})] \\ &= 2^n E[\delta_0(X(t_1))X'(t_1) \cdots \delta_0(X(t_n))X'(t_n) \cdot Y(\mathbf{s})] \\ &= 2^n E[X'(t_1) \cdots X'(t_n) \cdot Y(\mathbf{s}) \mid X(t_1) = \cdots = X(t_n) = 0] \cdot p_{\mathbf{t}}(\mathbf{0}), \end{aligned}$$

where $p_{\mathbf{t}}(\mathbf{0})$ is the density of the Gaussian vector $(X(t_1), \dots, X(t_n))$ at $(0, \dots, 0)$. Since $p_{\mathbf{t}}(\mathbf{0}) = (2\pi)^{-n/2}(\det K(\mathbf{t}))^{-1/2}$, the claim follows.

The above formal computation can be justified by using Watanabe's generalized Wiener functionals in the framework of Malliavin calculus over abstract Wiener spaces [22, 23]. \square

4.2 Conditional expectations

Recall the Gaussian power series f defined in (1.1). The process $\{f(t)\}_{-1 < t < 1}$ is centered real Gaussian with Cauchy covariance kernel

$$\sigma(s, t) = \frac{1}{1 - st}.$$

The following identity in law is a crucial property which the Gaussian process with Cauchy covariance kernel enjoys. Set

$$\mu(s, t) = \frac{s - t}{1 - st}.$$

Lemma 4.2. *For given $(t_1, t_2, \dots, t_n) \in \mathfrak{X}_n$, we have*

$$(f \mid f(t_1) = \cdots = f(t_n) = 0) \stackrel{\text{d}}{=} \mu(\cdot, \mathbf{t})f \quad (4.1)$$

where $\mu(s, \mathbf{t}) = \prod_{i=1}^n \mu(s, t_i)$. Moreover,

$$(f, f'(t_i), i = 1, \dots, n \mid f(t_1) = \cdots = f(t_n) = 0) \stackrel{\text{d}}{=} (\mu(\cdot, \mathbf{t})f, q_i(\mathbf{t})f(t_i), i = 1, \dots, n),$$

where $q_i(\mathbf{t}) = q_i(t_1, \dots, t_n)$ is defined in (3.2).

Proof. If a Gaussian process X has a covariance kernel $K(x, y)$, then that of the Gaussian process $(X \mid X(t) = 0)$, i.e. X given $X(t) = 0$, is equal to $K(x, y) - K(x, t)K(t, y)/K(t, t)$ whenever $K(t, t) > 0$. In the case of Cauchy kernel, we see that

$$\sigma(x, y) - \frac{\sigma(x, t)\sigma(t, y)}{\sigma(t, t)} = \mu(x, t)\mu(y, t)\sigma(x, y).$$

This implies that $(f|f(t) = 0) \stackrel{d}{=} \mu(\cdot, t)f$ as a process. Hence we obtain (4.1) by induction. As f' is a linear functional of f , we also have the identity in law as a Gaussian system

$$(f, f' \mid f(t_1) = \cdots = f(t_n) = 0) \stackrel{d}{=} (\mu(\cdot, \mathbf{t})f, \mu'(\cdot, \mathbf{t})f + \mu(\cdot, \mathbf{t})f').$$

Since $\mu(t_i, \mathbf{t}) = 0$ and $\mu'(t_i, \mathbf{t}) = q_i(\mathbf{t})$ for every $i = 1, 2, \dots, n$, we obtain the second equality in law. \square

4.3 Pfaffian expressions for derivations of signs

The following lemma is a conclusion of Lemmas 4.1 and 4.2.

Lemma 4.3. *Let $(\mathbf{t}, \mathbf{s}) = (t_1, t_2, \dots, t_n, s_1, s_2, \dots, s_m) \in \mathfrak{X}_{n+m}$. Then*

$$\begin{aligned} & \frac{\partial^n}{\partial t_1 \partial t_2 \cdots \partial t_n} E[\operatorname{sgn} f(t_1) \cdots \operatorname{sgn} f(t_n) \operatorname{sgn} f(s_1) \cdots \operatorname{sgn} f(s_m)] \\ &= (-1)^{n(n-1)/2} \prod_{i=1}^n \prod_{j=1}^m \operatorname{sgn}(s_j - t_i) \\ & \quad \times \left(\frac{2}{\pi} \right)^{n/2} (\det \Sigma(\mathbf{t}))^{1/2} E[f(t_1) \cdots f(t_n) \operatorname{sgn} f(s_1) \cdots \operatorname{sgn} f(s_m)], \end{aligned}$$

where $\Sigma(\mathbf{t}) = (\sigma(t_i, t_j))_{i,j=1}^n$.

Proof. From Lemma 4.2 and (3.3) we have

$$\begin{aligned} & E[f'(t_1) \cdots f'(t_n) \operatorname{sgn} f(s_1) \cdots \operatorname{sgn} f(s_m) \mid f(t_1) = \cdots = f(t_n) = 0] \\ &= \prod_{j=1}^m \operatorname{sgn} \mu(s_j, \mathbf{t}) \cdot \prod_{i=1}^n q_i(\mathbf{t}) \cdot E[f(t_1) \cdots f(t_n) \operatorname{sgn} f(s_1) \cdots \operatorname{sgn} f(s_m)] \\ &= \prod_{i=1}^n \prod_{j=1}^m \operatorname{sgn}(s_j - t_i) \cdot (-1)^{n(n-1)/2} \det \Sigma(\mathbf{t}) \cdot E[f(t_1) \cdots f(t_n) \operatorname{sgn} f(s_1) \cdots \operatorname{sgn} f(s_m)]. \end{aligned}$$

We have finished the proof by Lemma 4.1 with $X(t) = f(t)$. \square

Proposition 4.4. *For $\mathbf{t} = (t_1, \dots, t_{2n}) \in \mathfrak{X}_{2n}$, we have*

$$\begin{aligned} & \frac{\partial^{2n}}{\partial t_1 \partial t_2 \cdots \partial t_{2n}} E[\operatorname{sgn} f(t_1) \operatorname{sgn} f(t_2) \cdots \operatorname{sgn} f(t_{2n})] \\ &= \left(\frac{2}{\pi} \right)^n \prod_{1 \leq i < j \leq 2n} \operatorname{sgn}(t_j - t_i) \cdot \prod_{i=1}^{2n} \frac{1}{\sqrt{1 - t_i^2}} \cdot \operatorname{Pf} \left(\frac{t_i - t_j}{(1 - t_i t_j)^2} \right)_{i,j=1}^{2n}. \end{aligned} \quad (4.2)$$

Proof. Lemma 4.3 with $m = 0$ and with the replacement n by $2n$ gives

$$\frac{\partial^{2n}}{\partial t_1 \cdots \partial t_{2n}} E[\operatorname{sgn} f(t_1) \cdots \operatorname{sgn} f(t_{2n})] = (-1)^n \left(\frac{2}{\pi}\right)^n (\det \Sigma(\mathbf{t}))^{1/2} E[f(t_1) \cdots f(t_{2n})].$$

Here the Wick formula (3.6) gives

$$E[f(t_1) \cdots f(t_{2n})] = \operatorname{Hf}(E[f(t_i)f(t_j)])_{i,j=1}^{2n} = \operatorname{Hf}((1 - t_i t_j)^{-1})_{i,j=1}^{2n}$$

and Cauchy's determinant identity (3.1) gives

$$(-1)^n (\det \Sigma(\mathbf{t}))^{1/2} = \prod_{1 \leq i < j \leq 2n} \operatorname{sgn}(t_j - t_i) \cdot \prod_{i=1}^{2n} \frac{1}{\sqrt{1 - t_i^2}} \cdot \prod_{1 \leq i < j \leq 2n} \frac{t_i - t_j}{1 - t_i t_j}.$$

Hence we obtain

$$\begin{aligned} & \frac{\partial^{2n}}{\partial t_1 \partial t_2 \cdots \partial t_{2n}} E[\operatorname{sgn} f(t_1) \operatorname{sgn} f(t_2) \cdots \operatorname{sgn} f(t_{2n})] \\ &= \left(\frac{2}{\pi}\right)^n \prod_{1 \leq i < j \leq 2n} \operatorname{sgn}(t_j - t_i) \cdot \prod_{i=1}^{2n} \frac{1}{\sqrt{1 - t_i^2}} \cdot \prod_{1 \leq i < j \leq 2n} \frac{t_i - t_j}{1 - t_i t_j} \cdot \operatorname{Hf}\left(\frac{1}{1 - t_i t_j}\right)_{i,j=1}^{2n}. \end{aligned}$$

The desired Pfaffian expression follows from the Pfaffian-Hafnian identity (3.5). \square

5 Proof of Theorems 4 and 5

5.1 Proof of Theorem 5

Proposition 4.4 can be expressed as

$$\begin{aligned} & \frac{\partial^{2n}}{\partial t_1 \partial t_2 \cdots \partial t_{2n}} E[\operatorname{sgn} f(t_1) \operatorname{sgn} f(t_2) \cdots \operatorname{sgn} f(t_{2n})] \\ &= \left(\frac{2}{\pi}\right)^n \prod_{1 \leq i < j \leq 2n} \operatorname{sgn}(t_j - t_i) \cdot \operatorname{Pf}(\mathbb{K}_{11}(t_i, t_j))_{i,j=1}^{2n} \\ &= \left(\frac{2}{\pi}\right)^n \prod_{1 \leq i < j \leq 2n} \operatorname{sgn}(t_j - t_i) \cdot \frac{\partial^{2n}}{\partial t_1 \cdots \partial t_{2n}} \operatorname{Pf}(\mathbb{K}_{22}(t_i, t_j))_{i,j=1}^{2n}. \end{aligned}$$

If we remove the differential symbol $\frac{\partial^{2n}}{\partial t_1 \cdots \partial t_{2n}}$ in the above equation, then we get the equality on Theorem 5. The goal of the present subsection is to prove that this observation is veritably true.

For each subset set I of $\{1, 2, \dots, 2n\}$, we define the $2n \times 2n$ skew symmetric matrix $\mathbb{L}^I = \mathbb{L}^I(\mathbf{t})$, the (i, j) -entry of which is

$$\mathbb{L}_{ij}^I = \begin{cases} \mathbb{K}_{11}(t_i, t_j) = \frac{\partial^2}{\partial t_i \partial t_j} \mathbb{K}_{22}(t_i, t_j) & \text{if } i, j \in I, \\ \mathbb{K}_{12}(t_i, t_j) = \frac{\partial}{\partial t_i} \mathbb{K}_{22}(t_i, t_j) & \text{if } i \in I \text{ and } j \in I^c, \\ \mathbb{K}_{21}(t_i, t_j) = \frac{\partial}{\partial t_j} \mathbb{K}_{22}(t_i, t_j) & \text{if } i \in I^c \text{ and } j \in I, \\ \mathbb{K}_{22}(t_i, t_j) & \text{if } i, j \in I^c. \end{cases} \quad (5.1)$$

In particular, we put $\mathbb{L}^{[k]} = \mathbb{L}^I$ if $I = \{1, 2, \dots, k\}$ and $\mathbb{L}^{[0]} = \mathbb{L}^\emptyset$.

Lemma 5.1. *The following two claims hold true.*

1. For each $k = 0, 1, \dots, 2n - 1$, $\frac{\partial}{\partial t_{k+1}} \text{Pf } \mathbb{L}^{[k]} = \text{Pf } \mathbb{L}^{[k+1]}$.
2. $\text{Pf } \mathbb{L}^{[k]}$ is skew symmetric in $t_{k+1}, t_{k+2}, \dots, t_{2n}$.

Proof. Recall the definition of the Pfaffian

$$\frac{\partial}{\partial t_{k+1}} \text{Pf } \mathbb{L}^{[k]} = \sum_{\eta} (\text{sgn } \eta) \frac{\partial}{\partial t_{k+1}} \prod_{i=1}^n \mathbb{L}_{\eta(2i-1)\eta(2i)}^{[k]}.$$

For each $i < j$, we see that:

$$\begin{aligned} \text{if } j = k + 1, \text{ then } & \frac{\partial}{\partial t_{k+1}} \mathbb{L}_{i, k+1}^{[k]} = \frac{\partial}{\partial t_{k+1}} \mathbb{K}_{12}(t_i, t_{k+1}) = \mathbb{K}_{11}(t_i, t_{k+1}) = \mathbb{L}_{i, k+1}^{[k+1]}; \\ \text{if } i = k + 1, \text{ then } & \frac{\partial}{\partial t_{k+1}} \mathbb{L}_{k+1, j}^{[k]} = \frac{\partial}{\partial t_{k+1}} \mathbb{K}_{22}(t_{k+1}, t_j) = \mathbb{K}_{12}(t_{k+1}, t_j) = \mathbb{L}_{k+1, j}^{[k+1]}; \\ \text{if } i, j \neq k + 1, \text{ then } & \frac{\partial}{\partial t_{k+1}} \mathbb{L}_{i, j}^{[k]} = 0. \end{aligned}$$

Hence we have

$$\frac{\partial}{\partial t_{k+1}} \text{Pf } \mathbb{L}^{[k]} = \sum_{\eta} (\text{sgn } \eta) \prod_{i=1}^n \mathbb{L}_{\eta(2i-1)\eta(2i)}^{[k+1]} = \text{Pf } \mathbb{L}^{[k+1]},$$

which is the first claim.

Pfaffians are skew symmetric with respect to the change of the order of rows/columns, i.e., $\text{Pf}(a_{\eta(i)\eta(j)}) = (\text{sgn } \eta) \text{Pf } A$ for any $2n \times 2n$ skew symmetric matrix $A = (a_{ij})$ and a permutation η on $\{1, 2, \dots, 2n\}$. Hence the second claim follows from the definition of $\mathbb{L}^{[k]}$. \square

Put

$$\mathfrak{X}_{2n}^< = \{(t_1, \dots, t_{2n}) \in \mathfrak{X}_{2n} \mid t_1 < \dots < t_{2n}\}.$$

Lemma 5.2.

$$\lim_{\substack{t_{2n} \rightarrow 1 \\ (t_1, \dots, t_{2n}) \in \mathfrak{X}_{2n}^<}} E[\operatorname{sgn} f(t_1) \cdots \operatorname{sgn} f(t_{2n})] = 0.$$

Proof. If we put $X(t) = \sqrt{1-t^2}f(t)$, then

$$E[\operatorname{sgn} f(t_1) \cdots \operatorname{sgn} f(t_{2n})] = E[\operatorname{sgn} X(t_1) \cdots \operatorname{sgn} X(t_{2n})]$$

and $E[X(t_i)X(t_j)] = c(t_i, t_j) = \frac{\sqrt{(1-t_i^2)(1-t_j^2)}}{1-t_it_j}$. Furthermore, since $\lim_{t_{2n} \rightarrow 1} c(t_i, t_{2n}) = \delta_{i,2n}$, the random variable $X(t_{2n})$ converges in distribution to a standard Gaussian variable independent of other X_i ($i < 2n$), which means that $E[\operatorname{sgn} X(t_1) \cdots \operatorname{sgn} X(t_{2n})] \rightarrow 0$. \square

Lemma 5.3. For each $k = 0, 1, 2, \dots, 2n-1$,

$$\lim_{\substack{t_{2n} \rightarrow 1 \\ (t_1, \dots, t_{2n}) \in \mathfrak{X}_{2n}^<}} \operatorname{Pf} \mathbb{L}^{[k]} = 0.$$

Proof. Taking the limit $t_{2n} \rightarrow 1$, each entry in the last row and column of $\mathbb{L}^{[k]}$ converges to zero, and so does $\operatorname{Pf} \mathbb{L}^{[k]}$. \square

Lemma 5.4. Let $(t_1, \dots, t_{2n}) \in \mathfrak{X}_{2n}^<$. For each $k = 0, 1, \dots, 2n$,

$$\frac{\partial^k}{\partial t_1 \cdots \partial t_k} E[\operatorname{sgn} f(t_1) \cdots \operatorname{sgn} f(t_{2n})] = \left(\frac{2}{\pi}\right)^n \operatorname{Pf} \mathbb{L}^{[k]}. \quad (5.2)$$

Proof. Consider the function $Z^{[k]}$ on \mathfrak{X}_{2n} defined by

$$\begin{aligned} Z^{[k]}(t_1, \dots, t_{2n}) &= \frac{\partial^k}{\partial t_1 \cdots \partial t_k} E[\operatorname{sgn} f(t_1) \cdots \operatorname{sgn} f(t_{2n})] \\ &\quad - \left(\frac{2}{\pi}\right)^n \prod_{1 \leq i < j \leq 2n} \operatorname{sgn}(t_j - t_i) \cdot \operatorname{Pf} \mathbb{L}^{[k]}. \end{aligned}$$

Since $\mathbb{L}^{[2n]} = (\mathbb{K}_{11}(t_i, t_j))_{i,j=1}^{2n}$, Proposition 4.4 implies that $Z^{[2n]} \equiv 0$ on \mathfrak{X}_{2n} . Let $k \in \{0, 1, \dots, 2n-1\}$ and suppose that $Z^{[k+1]} \equiv 0$ on $\mathfrak{X}_{2n}^<$. Our goal is to prove $Z^{[k]} \equiv 0$ on $\mathfrak{X}_{2n}^<$.

From the first statement of Lemma 5.1, $\frac{\partial}{\partial t_{k+1}} Z^{[k]}(t_1, \dots, t_{2n}) = Z^{[k+1]}(t_1, \dots, t_{2n})$, and hence our assumption implies $\frac{\partial}{\partial t_{k+1}} Z^{[k]}(t_1, \dots, t_{2n}) = 0$. Therefore $Z^{[k]}$ is independent of t_{k+1} . From the second statement of Lemma 5.1, $Z^{[k]}$ is symmetric in t_{k+1}, \dots, t_{2n} , and therefore $Z^{[k]}$ is also independent of t_{2n} . However,

$$\lim_{\substack{t_{2n} \rightarrow 1 \\ (t_1, \dots, t_{2n}) \in \mathfrak{X}_{2n}^<}} Z^{[k]} = 0$$

by Lemmas 5.2 and 5.3. Hence, $Z^{[k]}$ must be identically zero on $\mathfrak{X}_{2n}^<$. \square

Proof of Theorem 5. Lemma 5.4 for $k = 0$ implies

$$E[\operatorname{sgn} f(t_1) \cdots \operatorname{sgn} f(t_{2n})] = \left(\frac{2}{\pi}\right)^n \operatorname{Pf}(\mathbb{K}_{22}(t_i, t_j))_{i,j=1}^{2n}$$

for $t_1 < \cdots < t_{2n}$. Since $\operatorname{Pf}(\mathbb{K}_{22}(t_i, t_j))_{i,j=1}^{2n}$ is skew symmetric in t_1, \dots, t_{2n} ,

$$\prod_{1 \leq i < j \leq 2n} \operatorname{sgn}(t_j - t_i) \cdot \operatorname{Pf}(\mathbb{K}_{22}(t_i, t_j))_{i,j=1}^{2n}$$

is symmetric and coincides with $E[\operatorname{sgn} f(t_1) \cdots \operatorname{sgn} f(t_{2n})]$ on \mathfrak{X}_{2n} . Thus we have obtained Theorem 5. \square

The following corollary is a conclusion of Theorem 5.

Corollary 7. For $(t_1, \dots, t_{2n}) \in \mathfrak{X}_{2n}$ and a subset $I = \{i_1 < i_2 < \cdots < i_k\}$ in $\{1, 2, \dots, 2n\}$,

$$\frac{\partial^k}{\partial t_{i_1} \cdots \partial t_{i_k}} E[\operatorname{sgn} f(t_1) \cdots \operatorname{sgn} f(t_{2n})] = \left(\frac{2}{\pi}\right)^n \prod_{1 \leq i < j \leq 2n} \operatorname{sgn}(t_j - t_i) \cdot \operatorname{Pf} \mathbb{L}^I,$$

where \mathbb{L}^I is defined in (5.1).

Proof. Observe that $\frac{\partial^k}{\partial t_{i_1} \cdots \partial t_{i_k}} \operatorname{Pf} \mathbb{L}^\emptyset = \operatorname{Pf} \mathbb{L}^I$. \square

5.2 Proof of Theorem 4

Lemma 5.5. For $(t_1, \dots, t_n) \in \mathfrak{X}_n$,

$$\begin{aligned} & \lim_{s_1 \rightarrow t_1 + 0} \cdots \lim_{s_n \rightarrow t_n + 0} \frac{\partial^n}{\partial t_1 \cdots \partial t_n} E[\operatorname{sgn} f(t_1) \cdots \operatorname{sgn} f(t_n) \operatorname{sgn} f(s_1) \cdots \operatorname{sgn} f(s_n)] \\ &= \left(\frac{2}{\pi}\right)^n \operatorname{Pf}(\mathbb{K}(t_i, t_j))_{i,j=1}^n. \end{aligned}$$

Proof. Take $-1 < t_1 < s_1 < t_2 < s_2 < \cdots < t_n < s_n < 1$. Corollary 7 with $I = \{1, 3, 5, \dots, 2n-1\}$ and with the replacement (t_1, \dots, t_{2n}) by $(t_1, s_1, \dots, t_n, s_n)$ gives

$$\begin{aligned} & \frac{\partial^n}{\partial t_1 \cdots \partial t_n} E[\operatorname{sgn} f(t_1) \cdots \operatorname{sgn} f(t_n) \operatorname{sgn} f(s_1) \cdots \operatorname{sgn} f(s_n)] \\ &= \left(\frac{2}{\pi}\right)^n \operatorname{Pf} \begin{pmatrix} \mathbb{K}_{11}(t_i, t_j) & \mathbb{K}_{12}(t_i, s_j) \\ \mathbb{K}_{21}(s_i, t_j) & \mathbb{K}_{22}(s_i, s_j) \end{pmatrix}_{i,j=1}^n. \end{aligned}$$

Taking the limit $s_1 \rightarrow t_1, \dots, s_n \rightarrow t_n$, it converges to $\left(\frac{2}{\pi}\right)^n \operatorname{Pf}(\mathbb{K}(t_i, t_j))_{i,j=1}^n$ for $t_1 < \cdots < t_n$. From the symmetry for t_1, \dots, t_n , the achieved result holds true for every $(t_1, \dots, t_n) \in \mathfrak{X}_n$. \square

Proof of Theorem 4. We use Lemma 4.3 with $m = n$. The identity in the lemma holds true for $-1 < t_1 < s_1 < t_2 < s_2 < \dots < t_n < s_n < 1$. Note that $\prod_{i=1}^n \prod_{j=1}^n \operatorname{sgn}(s_j - t_i) = (-1)^{n(n-1)/2}$. Taking the limit $s_1 \rightarrow t_1, \dots, s_n \rightarrow t_n$,

$$\begin{aligned} & \lim_{s_1 \rightarrow t_1 + 0} \dots \lim_{s_n \rightarrow t_n + 0} \frac{\partial^n}{\partial t_1 \dots \partial t_n} E[\operatorname{sgn} f(t_1) \dots \operatorname{sgn} f(t_n) \operatorname{sgn} f(s_1) \dots \operatorname{sgn} f(s_n)] \\ &= \left(\frac{2}{\pi}\right)^{n/2} (\det \Sigma(\mathbf{t}))^{1/2} E[|f(t_1) \dots f(t_n)|] \end{aligned}$$

for $-1 < t_1 < \dots < t_n < 1$. From the symmetry for t_1, \dots, t_n , the above equation holds true for every $(t_1, \dots, t_n) \in \mathfrak{X}_n$. Combining this fact with Lemma 5.5, we obtain Theorem 4. \square

6 Proofs of Theorem 1, Corollary 2 and Corollary 3

6.1 Proof of Theorem 1

Hammersley's formula describes correlation functions of zeros of random polynomials, which was observed by Hammersley [8] and it is extended to Gaussian analytic functions as Corollary 3.4.2 in [9]. The following lemma is a real version of Hammersley's formula for correlation functions of Gaussian analytic functions.

Lemma 6.1. *Let $X(t)$ be a random power series with independent real Gaussian coefficients defined on an interval $(-1, 1)$ with covariance kernel K . If $\det K(\mathbf{t}) = \det(K(t_i, t_j))_{i,j=1}^n$ does not vanish anywhere on \mathfrak{X}_n , then the n -point correlation function for real zeros of f exists and is given by*

$$\rho_n(t_1, \dots, t_n) = \frac{E[|X'(t_1) \dots X'(t_n)| \mid X(t_1) = \dots = X(t_n) = 0]}{(2\pi)^{n/2} \sqrt{\det K(\mathbf{t})}}$$

for $\mathbf{t} = (t_1, \dots, t_n) \in \mathfrak{X}_n$.

Proof. This can be proved in almost the same way as in the proof of (3.4.1) in Corollary 3.4.2 in [9]. The only difference is that the exponent of $|X'(t_1) \dots X'(t_n)|$ is 1 in the case of real Gaussian coefficients instead of 2 in the complex case. This is due to the fact that the Jacobian determinant of $F(\mathbf{t}) = (X(t_1), \dots, X(t_n))$ is equal to $|X'(t_1) \dots X'(t_n)|$ when X is a real-valued differentiable function while $|X'(t_1) \dots X'(t_n)|^2$ when X is complex-valued. \square

Proof of Theorem 1. From Lemma 4.2 and (3.3) we have

$$\begin{aligned} & E[|f'(t_1) \dots f'(t_n)| \mid f(t_1) = \dots = f(t_n) = 0] \\ &= |q_1(\mathbf{t}) \dots q_n(\mathbf{t})| E[|f(t_1) \dots f(t_n)|] \\ &= \det \Sigma(\mathbf{t}) \cdot E[|f(t_1) \dots f(t_n)|], \end{aligned}$$

and it follows from Lemma 6.1 that

$$\rho_n(t_1, \dots, t_n) = (2\pi)^{-n/2} (\det \Sigma(\mathbf{t}))^{1/2} E[|f(t_1) \cdots f(t_n)|].$$

Hence Theorem 1 follows from Theorem 4. \square

6.2 Proof of Corollaries 2 and 3

Proof of Corollary 2. From (2.4) we observe that

$$R(s, t) = 1 + |\mu(s, t)|c(s, t) \arcsin c(s, t) - c(s, t)^2,$$

and so $R(s, s) = 0$ and $R(s, \pm 1) = 1$. A simple calculation yields

$$\begin{aligned} \frac{\partial}{\partial t} |\mu(s, t)| &= \frac{\operatorname{sgn}(t-s)}{1-t^2} c(s, t)^2, & \frac{\partial}{\partial t} c(s, t) &= -\frac{\operatorname{sgn}(t-s)}{1-t^2} |\mu(s, t)|c(s, t), \\ \frac{\partial}{\partial t} \arcsin c(s, t) &= -\frac{\operatorname{sgn}(t-s)}{1-t^2} c(s, t) \end{aligned}$$

and hence we obtain

$$\frac{\partial}{\partial t} R(s, t) = \frac{\operatorname{sgn}(t-s)}{1-t^2} c(s, t)g(s, t),$$

where

$$g(s, t) := \{c(s, t)^2 \arcsin c(s, t) - |\mu(s, t)|^2 \arcsin c(s, t) + c(s, t)|\mu(s, t)|\}.$$

Since $g(s, \pm 1) = 0$ and

$$\frac{\partial}{\partial t} g(s, t) = \frac{-4 \operatorname{sgn}(t-s)}{1-t^2} |\mu(s, t)|c(s, t)^2 \arcsin c(s, t),$$

we have $g(s, t) \geq 0$. This implies the claim. \square

Proof of Corollary 3. The first equality immediately follows from $EN_r = \int_{-r}^r \rho_1(s)ds$. Recall that

$$\operatorname{Var} N_r = \int_{-r}^r \int_{-r}^r \rho_2(s, t)dsdt + \int_{-r}^r \rho_1(s)ds - \left(\int_{-r}^r \rho_1(s)ds \right)^2.$$

We recall 2-correlation function

$$\rho_2(s, t) = \pi^{-2} \{ \mathbb{K}_{12}(s, s)\mathbb{K}_{12}(t, t) - \mathbb{K}_{11}(s, t)\mathbb{K}_{22}(s, t) + \mathbb{K}_{12}(s, t)\mathbb{K}_{21}(s, t) \}$$

as in (2.4). Taking the discontinuity of $\mathbb{K}_{22}(s, t)$ at $s = t$ into account and using integration by parts together with (2.3), we have

$$\begin{aligned}
& \int_{-r}^r \mathbb{K}_{11}(s, t) \mathbb{K}_{22}(s, t) dt \\
&= \int_{-r}^s \frac{\partial^2 \mathbb{K}_{22}}{\partial s \partial t}(s, t) \mathbb{K}_{22}(s, t) dt + \int_s^r \frac{\partial^2 \mathbb{K}_{22}}{\partial s \partial t}(s, t) \mathbb{K}_{22}(s, t) dt \\
&= \left[\frac{\partial \mathbb{K}_{22}}{\partial s}(s, t) \mathbb{K}_{22}(s, t) \right]_{t=-r}^{s=0} + \left[\frac{\partial \mathbb{K}_{22}}{\partial s}(s, t) \mathbb{K}_{22}(s, t) \right]_{t=s+0}^r - \int_{-r}^r \mathbb{K}_{12}(s, t) \mathbb{K}_{21}(s, t) dt \\
&= \{ \mathbb{K}_{12}(s, r) \mathbb{K}_{22}(s, r) - \mathbb{K}_{12}(s, -r) \mathbb{K}_{22}(s, -r) - \pi \mathbb{K}_{12}(s, s) \} - \int_{-r}^r \mathbb{K}_{12}(s, t) \mathbb{K}_{21}(s, t) dt.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
& \int_{-r}^r ds \{ \mathbb{K}_{12}(s, r) \mathbb{K}_{22}(s, r) - \mathbb{K}_{12}(s, -r) \mathbb{K}_{22}(s, -r) \} \\
&= \frac{1}{2} [\mathbb{K}_{22}(s, r)^2 - \mathbb{K}_{22}(s, -r)^2]_{-r}^r = \mathbb{K}_{22}(r, r)^2 - \mathbb{K}_{22}(r, -r)^2 = O(1).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\text{Var } N_r &= 2\pi^{-2} \left(\pi \int_{-r}^r \mathbb{K}_{12}(s, s) ds + \int_{-r}^r \int_{-r}^r \mathbb{K}_{12}(s, t) \mathbb{K}_{21}(s, t) ds dt \right) + O(1) \\
&= 2\pi^{-2} \left(\pi \log \frac{1+r}{1-r} - 2 \log \frac{1+r^2}{1-r^2} \right) + O(1).
\end{aligned}$$

This implies the assertion. \square

7 Proof of Theorem 6

7.1 Complex-valued Gaussian processes

In this section, we assume that a process $X = \{X(\lambda)\}_{\lambda \in \Lambda}$ is centered, i.e., $E[X(\lambda)] = 0$ for each $\lambda \in \Lambda$. Let $X = \{X(\lambda)\}_{\lambda \in \Lambda}$ be a complex-valued Gaussian process in the sense that the real and imaginary parts form a real Gaussian process. We say that a complex-valued Gaussian process is a *complex Gaussian process* if the real and imaginary parts are mutually independent and have the same variance.

For a complex-valued Gaussian process X , we use three 2×2 matrices

$$\begin{aligned}\mathbb{M}(\lambda, \mu) &= \mathbb{M}_X(\lambda, \mu) = \begin{pmatrix} E[X(\lambda)\overline{X(\mu)}] & E[X(\lambda)X(\mu)] \\ E[\overline{X(\lambda)}X(\mu)] & E[\overline{X(\lambda)}\overline{X(\mu)}] \end{pmatrix}, \\ \widehat{\mathbb{M}}(\lambda, \mu) &= \widehat{\mathbb{M}}_X(\lambda, \mu) = \begin{pmatrix} E[X(\lambda)X(\mu)] & E[X(\lambda)\overline{X(\mu)}] \\ E[\overline{X(\lambda)}X(\mu)] & E[\overline{X(\lambda)}\overline{X(\mu)}] \end{pmatrix}, \\ \widetilde{\mathbb{M}}(\lambda, \mu) &= \widetilde{\mathbb{M}}_X(\lambda, \mu) = \begin{pmatrix} E[\Re X(\lambda)\Re X(\mu)] & E[\Re X(\lambda)\Im X(\mu)] \\ E[\Im X(\lambda)\Re X(\mu)] & E[\Im X(\lambda)\Im X(\mu)] \end{pmatrix}.\end{aligned}$$

For $\lambda_1, \lambda_2, \dots, \lambda_n \in \Lambda$, the matrix $(\mathbb{M}(\lambda_i, \lambda_j))_{i,j=1}^n$ is Hermitian, $(\widehat{\mathbb{M}}(\lambda_i, \lambda_j))_{i,j=1}^n$ is complex symmetric, and $(\widetilde{\mathbb{M}}(\lambda_i, \lambda_j))_{i,j=1}^n$ is real symmetric. The real Gaussian vector

$$(\Re X(\lambda_1), \Im X(\lambda_1), \dots, \Re X(\lambda_n), \Im X(\lambda_n))$$

has the covariance matrix $(\widetilde{\mathbb{M}}(\lambda_i, \lambda_j))_{i,j=1}^n$. It is easy to see that

$$\widetilde{\mathbb{M}}(\lambda, \mu) = \frac{1}{4}U\mathbb{M}(\lambda, \mu)U^*, \quad U = \begin{pmatrix} 1 & 1 \\ -\sqrt{-1} & \sqrt{-1} \end{pmatrix}. \quad (7.1)$$

A (centered) complex-valued Gaussian process is uniquely determined by \mathbb{M} or $\widehat{\mathbb{M}}$.

Lemma 7.1. *For $\lambda_1, \dots, \lambda_n \in \Lambda$,*

$$\mathbb{E}[|X(\lambda_1) \cdots X(\lambda_n)|^2] = \text{Hf}(\widehat{\mathbb{M}}(\lambda_i, \lambda_j))_{i,j=1}^n. \quad (7.2)$$

Proof. Let $Y_a(\lambda) = \Re X(\lambda) + a\Im X(\lambda)$ for $a \in \mathbb{R}$. It follows from the Wick formula (3.6) that

$$\mathbb{E}[Y_a(\lambda_1)Y_b(\lambda_1) \cdots Y_a(\lambda_n)Y_b(\lambda_n)] = \text{Hf} \begin{pmatrix} E[Y_a(\lambda_i)Y_a(\lambda_j)] & E[Y_a(\lambda_i)Y_b(\lambda_j)] \\ E[Y_b(\lambda_i)Y_a(\lambda_j)] & E[Y_b(\lambda_i)Y_b(\lambda_j)] \end{pmatrix}_{i,j=1}^n. \quad (7.3)$$

By analytic continuation, the formula (7.3) still holds for $a, b \in \mathbb{C}$. Therefore, by setting $a = -b = \sqrt{-1}$, we obtain the result. \square

7.2 Conditional expectations for complex cases

Throughout this section, we use the following notation. Put

$$\mu(z, w) = \frac{z - w}{1 - zw}.$$

Let $\mathbb{D}_+ = \{z \in \mathbb{D} \mid \Im z > 0\}$ and $z_1, \dots, z_n \in \mathbb{D}_+$. Moreover, we set as $z_{j+n} := \overline{z_j}$, $j = 1, 2, \dots, n$, and define $q_i(\mathbf{z}) := q_i(z_1, \dots, z_{2n})$ by (3.2) for $i = 1, 2, \dots, 2n$:

$$q_i(\mathbf{z}) = \frac{1}{1 - z_i^2} \prod_{\substack{1 \leq k \leq 2n \\ k \neq i}} \frac{z_i - z_k}{1 - z_i z_k}.$$

Let f be the Gaussian power series defined by (1.1). Then $\{f(z)\}_{z \in \mathbb{D}_+}$ is a complex-valued Gaussian process with

$$\mathbb{M}_f(z, w) = \begin{pmatrix} \frac{1}{1-z\bar{w}} & \frac{1}{1-zw} \\ \frac{1}{1-\bar{z}w} & \frac{1}{1-\bar{z}\bar{w}} \end{pmatrix}.$$

Lemma 7.2. For $\eta \in \mathbb{D}_+$,

$$(f \mid f(\eta) = 0) \stackrel{d}{=} \mu(\cdot, \eta) \mu(\cdot, \bar{\eta}) f. \quad (7.4)$$

Moreover,

$$(f'(z_i), i = 1, 2, \dots, n \mid f(z_1) = \dots = f(z_n) = 0) \stackrel{d}{=} (q_i(\mathbf{z}) f(z_i), i = 1, 2, \dots, n). \quad (7.5)$$

Proof. The real Gaussian vector $(\Re f(z), \Im f(w))$ given $\Re f(\eta) = \Im f(\eta) = 0$ has the covariance matrix

$$\begin{aligned} & \widetilde{\mathbb{M}}_f(z, w) - \widetilde{\mathbb{M}}_f(z, \eta) \widetilde{\mathbb{M}}_f(\eta, \eta)^{-1} \widetilde{\mathbb{M}}_f(\eta, w) \\ &= \frac{1}{4} U [\mathbb{M}_f(z, w) - \mathbb{M}_f(z, \eta) \mathbb{M}_f(\eta, \eta)^{-1} \mathbb{M}_f(\eta, w)] U^* \end{aligned}$$

by (7.1). A direct computation gives

$$\mathbb{M}_f(z, w) - \mathbb{M}_f(z, \eta) \mathbb{M}_f(\eta, \eta)^{-1} \mathbb{M}_f(\eta, w) = \mathbb{M}_{Y_\eta}(z, w)$$

with $Y_\eta(z) = \mu(z, \eta) \mu(z, \bar{\eta}) f(z)$, and we obtain (7.4). The remaining statement follows from (7.4) in a manner similar to the proof of Lemma 4.2. \square

7.3 Correlation functions for complex zeros

We finally compute the correlation function $\rho_n^c(z_1, \dots, z_n)$ for complex zeros of f . Our starting point is the following Hammersley's formula (complex version), see [9] and compare with Lemma 6.1:

$$\rho_n^c(z_1, \dots, z_n) = \frac{E[|f'(z_1) \cdots f'(z_n)|^2 \mid f(z_1) = \dots = f(z_n) = 0]}{(2\pi)^n \sqrt{\det(\widetilde{\mathbb{M}}_f(z_i, z_j))_{i,j=1}^n}}. \quad (7.6)$$

Note that $(2\pi)^{-n} [\det(\widetilde{\mathbb{M}}_f(z_i, z_j))]^{-1/2}$ is the density of the real Gaussian vector

$$(\Re f(z_1), \Im f(z_1), \dots, \Re f(z_n), \Im f(z_n))$$

at $(0, 0, \dots, 0, 0)$.

Proposition 7.3. *Let*

$$M(\mathbf{z}) = \left(\frac{1}{1 - z_i \bar{z}_j} \right)_{i,j=1}^{2n} \quad \text{and} \quad \hat{M}(\mathbf{z}) = \left(\frac{1}{1 - z_i z_j} \right)_{i,j=1}^{2n}.$$

Then

$$\rho_n^c(z_1, \dots, z_n) = \frac{(-1)^n \det \hat{M}(\mathbf{z}) \cdot \text{Hf } \hat{M}(\mathbf{z})}{\pi^n \sqrt{\det M(\mathbf{z})}}.$$

Proof. Let us compute the numerator on (7.6). Equation (7.5) gives

$$E[|f'(z_1) \cdots f'(z_n)|^2 \mid f(z_1) = \cdots = f(z_n) = 0] = |q_1(\mathbf{z}) \cdots q_n(\mathbf{z})|^2 E[|f(z_1) \cdots f(z_n)|^2].$$

Here, since $\overline{q_j(\mathbf{z})} = q_{j+n}(\mathbf{z})$ for $j = 1, 2, \dots, n$, it follows from (3.3) that

$$|q_1(\mathbf{z}) \cdots q_n(\mathbf{z})|^2 = \prod_{i=1}^{2n} q_i(\mathbf{z}) = (-1)^n \det \hat{M}(\mathbf{z}).$$

Furthermore, from (7.2) we have

$$E[|f(z_1) \cdots f(z_n)|^2] = \text{Hf}(\widehat{\mathbb{M}}_f(z_i, z_j))_{i,j=1}^n = \text{Hf } \hat{M}(\mathbf{z}).$$

On the other hand, the denominator on (7.6) is computed by using (7.1):

$$\det(\widetilde{\mathbb{M}}_f(z_i, z_j))_{i,j=1}^n = 4^{-n} \det M(\mathbf{z}).$$

Consequently, we obtain the result from (7.6). \square

Proof of Theorem 6. By the Cauchy determinant formula (3.1),

$$\begin{aligned} \det \hat{M}(\mathbf{z}) &= \prod_{i=1}^{2n} \frac{1}{1 - z_i^2} \prod_{1 \leq i < j \leq 2n} \left(\frac{z_i - z_j}{1 - z_i z_j} \right)^2, \\ \sqrt{\det M(\mathbf{z})} &= \prod_{i=1}^{2n} \frac{1}{\sqrt{1 - |z_i|^2}} \cdot \prod_{1 \leq i < j \leq 2n} \left| \frac{z_i - z_j}{1 - z_i \bar{z}_j} \right|. \end{aligned}$$

By noting that $z_{i+n} = \bar{z}_i$ ($i = 1, 2, \dots, n$), it is easy to see that

$$\frac{\det \hat{M}(\mathbf{z})}{\sqrt{\det M(\mathbf{z})}} = (-1)^{n(n-1)/2} \prod_{i=1}^n \frac{1}{|1 - z_i^2|} \cdot \prod_{i=1}^n \frac{z_i - \bar{z}_i}{|z_i - \bar{z}_i|} \left(\prod_{1 \leq i < j \leq 2n} \frac{z_i - z_j}{1 - z_i z_j} \right).$$

Since $\frac{z - \bar{z}}{|z - \bar{z}|} = \sqrt{-1}$ for $\Im z > 0$, from Proposition 7.3 and Pfaffian-Hafnian identity (3.5) we see that

$$\rho_n^c(z_1, \dots, z_n) = \frac{(-1)^{n(n-1)/2}}{(\pi \sqrt{-1})^n} \prod_{i=1}^n \frac{1}{|1 - z_i^2|} \cdot \text{Pf} \left(\frac{z_i - z_j}{(1 - z_i z_j)^2} \right)_{i,j=1}^{2n}.$$

By changing rows and columns, we finally obtain

$$\rho_n^c(z_1, \dots, z_n) = \frac{1}{(\pi\sqrt{-1})^n} \prod_{i=1}^n \frac{1}{|1 - z_i^2|} \cdot \text{Pf}(\mathbb{K}^c(z_i, z_j))_{i,j=1}^n.$$

□

Acknowledgments

The first author (SM)'s work was supported by JSPS Grant-in-Aid for Young Scientists (B) 22740060. The second author (TS)'s work was supported in part by JSPS Grant-in-Aid for Scientific Research (B) 22340020. S.M. would like to thank Yuzuru Inahama for his helpful conversations.

References

- [1] H. Bohr and B. Jessen, Über die Werteverteilung der Riemannschen Zetafunktion, *Acta Mathematica* **54** (1930), 1–35.
- [2] H. Bohr and B. Jessen, Über die Werteverteilung der Riemannschen Zetafunktion, *Acta Mathematica* **58** (1932), 1–55.
- [3] A. Borodin and C. D. Sinclair, The Ginibre ensemble of real random matrices and its scaling limits, *Comm. Math. Phys.* **291** (2009), no. 1, 177–224.
- [4] T. Ceccherini-Silberstein, F. Scarabotti, and F. Tolli, Representation theory of the symmetric groups. The Okounkov-Vershik approach, character formulas, and partition algebras, *Cambridge Studies in Advanced Mathematics*, 121. Cambridge University Press, 2010.
- [5] A. Edelman and E. Kostlan, How many zeros of a random polynomial are real?, *Bull. Amer. Math. Soc.* **32** (1995), 1–37.
- [6] P. J. Forrester, The limiting Kac random polynomial and truncated random orthogonal matrices, available at [arXiv:1009.3066v1](https://arxiv.org/abs/1009.3066)
- [7] P. J. Forrester and T. Nagao, Eigenvalue statistics of the real Ginibre ensemble, *Phys. Rev. Lett.* **99**, (2007), 050603, 4 pp.
- [8] J. M. Hammersley, The zeros of a random polynomial, *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955*, vol. II, pp. 89–111. University of California Press, Berkeley and Los Angeles, 1956.

- [9] J. B. Hough, M. Krishnapur, Y. Peres and B. Virág, Zeros of Gaussian Analytic Functions and Determinantal Point Processes, University Lecture Series, 51. American Mathematical Society, Providence, RI, 2009.
- [10] M. Ishikawa, H. Kawamuko, and S. Okada, A Pfaffian-Hafnian analogue of Borchartd's identity, *Electron. J. Combin.* **12** (2005), Note 9, 8 pp. (electronic).
- [11] M. Kac, On the average number of real roots of a random algebraic equation, *Bull. Amer. Math. Soc.* **49** (1943), 314–320.
- [12] J. P. Kahane, Some random series of functions, Heath, Lexington, 1968.
- [13] M. Krishnapur, From random matrices to random analytic functions, *Ann. Probab.* **37** (2009), 314–346.
- [14] B. F. Logan and L. A. Shepp, Real zeros of random polynomials. II, *Proc. London Math. Soc.* **18** (1968), 308–314.
- [15] S. Nabeya, Absolute moments in 2-dimensional normal distribution, *Ann. Inst. Statist. Math. Tokyo* **3** (1951), 1–6.
- [16] S. Nabeya, Absolute moments in 3-dimensional normal distribution, *Ann. Inst. Statist. Math. Tokyo* **4** (1952), 15–30.
- [17] Y. Peres and B. Virág, Zeros of the i.i.d. Gaussian power series: a conformally invariant determinantal process, *Acta Math.* **194** (2005), 1–35.
- [18] R. Paley and N. Wiener, Fourier transforms in the complex domain. Reprint of the 1934 original. American Mathematical Society Colloquium Publications, 19. American Mathematical Society, Providence, RI, 1987.
- [19] S. O. Rice, Mathematical theory of random noise, *Bell. System Tech. J.* **25** (1945), 46–156.
- [20] L. A. Shepp and R. J. Vanderbei, The complex zeros of random polynomials, *Trans. Amer. Math. Soc.* **347** (1995), 4365–4384.
- [21] R. Tribe and O. Zaboronski, Pfaffian formulae for one dimensional coalescing and annihilating systems, *Electronic Journal of Probability*, **16**, Article 76 (2011).
- [22] S. Watanabe, Lectures on stochastic differential equations and Malliavin's calculus, Tata Inst. of Fundamental research, Springer, 1984.
- [23] S. Watanabe, Analysis of Wiener Functionals (Malliavin Calculus) and its Applications to Heat Kernels, *Ann. Probab.* **15** (1987), 1–39.

- [24] A. Zvonkin, A, Matrix integrals and map enumeration: an accessible introduction, Combinatorics and physics (Marseilles, 1995), Math. Comput. Modelling 26 (1997), no. 8–10, 281–304.

SHO MATSUMOTO

Graduate School of Mathematics, Nagoya University, Nagoya, 464-8602, Japan.

`sho-matsumoto@math.nagoya-u.ac.jp`

TOMOYUKI SHIRAI

Institute of Mathematics for Industry, Kyushu University, Fukuoka 819-0395, Japan.

`shirai@imi.kyushu-u.ac.jp`